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GRID MANIFOLDS

S. A. ROBERTSON

This paper is intended to correlate the work of several authors on Riemannian manifolds on which there are defined systems of parallel fields of tangent planes. The main facts in this area are the reducibility theorem of de Rham [8] and the fibring theorems of Walker [14]. In order to make the exposition as simple as possible, we introduce the concepts of *local-product* and *grid*. Our principal result includes a 'grid' version of de Rham's theorem.

1. Almost-product and local-product structures

Let M be a smooth (i.e., C^∞) m-manifold without boundary. An *almost-product-structure* (or *AP-structure*) *of type* $(m_1, \cdots, m_r)$ on M is a direct sum decomposition $\tau_M = \bigoplus_{i=1}^{r} \xi_i$ of the tangent bundle of M into smooth subbundles ξ_i of dimension $m_i > 0$. Equivalently, an AP-structure on M is an ordered set $(\phi_1, \cdots, \phi_r)$ of mutually transverse smooth fields ϕ_i of tangent m_i-planes on M, with $\sum m_i = m$ (see [16]). Thus the m_i-plane $\phi_i(x)$ is the fibre of ξ_i at $x \in M$.

If each tangent field ϕ_i of an AP-structure $\Phi = (\phi_1, \cdots, \phi_r)$ on M is integrable, then we call Φ a *local-product structure*[1] (or *LP-structure*) on M *of type* $(m_1, \cdots, m_r)$, where as above $\dim \phi_i = m_i$. Equivalently, an LP-structure is an ordered set $\mathscr{F} = (\mathscr{F}_1, \cdots, \mathscr{F}_r)$ of mutually transverse smooth foliations $\mathscr{F}_i$ with foil-dimension $m_i > 0$, and with $\sum m_i = m$. We denote the foil of $\mathscr{F}_i$ through $x \in M$ by $F_i(x)$, and call the ordered set $F(x) = (F_1(x), \cdots, F_r(x))$ the *foil-sequence* of $\mathscr{F}$ at x.

If Φ is an AP-structure on M, then the pair (M, Φ) is called an *AP-manifold*. Likewise $(M, \mathscr{F})$ is an *LP-manifold*, where $\mathscr{F}$ is an LP-structure on M. Suppose then that (M, Φ) and (N, Ψ) are AP-manifolds of types $(m_1, \cdots, m_r)$ and $(n_1, \cdots, n_s)$ respectively. An *AP-morphism* from (M, Φ) to (M, Ψ) is a pair (f, ζ), where $f\colon M \to N$ is a smooth map, $\zeta\colon (1, \cdots, r) \to (1, \cdots, s)$ is order-preserving ((i.e., $i \le j \Rightarrow \zeta(i) \le \zeta(j)$), and for all $i = 1, \cdots, r$, and all $x \in M$, $Tf(\phi_i(x)) \subset \phi_j(f(x))$, where $j = \zeta(i)$. An LP-morphism from one LP-

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[1] This use of the term 'local-product structure' is slightly different from the usual one in that we do not require that arbitrary sums of the fields ϕ_i be integrable. Thus our concept might be more precisely termed a weak local product.

manifold $(M, \mathscr{F})$ to another $(N, \mathscr{G})$ is an AP-morphism from $(M, \mathscr{F})$ to $(N, \mathscr{G})$ as AP-manifolds.

We have now constructed the AP-category and the LP-category. Before proceeding further, we give some simple examples of LP-manifolds which will help to illuminate subsequent discussion.

(1) Let $\mathscr{F}_1$ be the foliation on M whose only foil is M itself. Then $(M, \mathscr{F})$ is an LP-manifold of type (m), there $\mathscr{F} = (\mathscr{F}_1)$.

(2) If $M_1, \cdots, M_r$ are smooth manifolds with $\dim M_i = m_i$, then $(M, \mathscr{F})$ is an LP-manifold of type $(m_1, \cdots, m_r)$, where $M = \times_{i=1}^{r} M_i$ and $F = (\mathscr{F}_i, \cdots, \mathscr{F}_r)$ is the ordered set of foliations for which $F_i(x) = \{x_1\} \times \cdots \times M_i \times \cdots \times \{x_r\}$. We call $\mathscr{F}$ the *standard* LP-structure on $\times M_i$.

(3) Let $\mathscr{F}_1$ be a smooth foliation of M, of codimension 1. Then any Riemannian metric on M determines a unique orthogonal foliation $\mathscr{F}_2$ of M of codimension $(m-1)$. Thus $(\mathscr{F}_1, \mathscr{F}_2)$ is an LP-structure of type $(m-1, 1)$ on M. An important special case of $\mathscr{F}_1$ is the Reeb foliation ([3], [7]) of the real 3-sphere S^3 by 2-dimensional foils. Another useful example of this construction may be described as follows.

(4) Let the Euclidean plane $\boldsymbol{R}^2$ be foliated by smooth curves as indicated by the solid lines in Figure 1. This foliation $\mathscr{F}_1$, together with its orthogonal complement $\mathscr{F}_2$ (indicated by broken lines in the Figure), determines an LP-structure $(\mathscr{F}_1, \mathscr{F}_2)$ of type $(1, 1)$ on $\boldsymbol{R}^2$.

(5) Let M be a parallelisable m-manifold, and let $X_1, \cdots, X_m$ be linearly independent smooth vector fields on M. Then any partition $[m_1, \cdots, m_r]$ of m determines an AP-structure $\Phi = (\phi_1, \cdots, \phi_n)$ on M, the field ϕ_i being generated by $\{X_a, \cdots, X_{a+s}\}$, where $a = m_{i-1}$ and $s = m_i$. However, Φ need not be an LP-structure except when $m_i = 1$ (all $i = 1, \cdots, r$).

(6) An important special case of (5) is obtained by taking $M = T^m = \times^m S^1$, where S^1 denotes the reals modulo 1. Any system of m mutually transverse (maximal) families of parallel straight lines in $\boldsymbol{R}^m$ determines an LP-structure of type $(1, \cdots, 1)$ on T^m.

We remark that if $(f, \zeta)\colon (M, \mathscr{F}) \to (N, \mathscr{G})$ is an LP-isomorphism, then $f\colon M \to N$ is necessarily a diffeomorphism and $\mathscr{F}, \mathscr{G}$ are of the same type. The converse statement, however, is false. For instance, Example (6) includes infinitely many LP-manifolds $(T^m, \mathscr{F})$ of type $(1, \cdots, 1)$ which are not LP-isomorphic. Again, the LP-manifold $(\boldsymbol{R}^2, \mathscr{F})$ of Example (4) and the standard LP-manifold $(\boldsymbol{R}^2, \mathscr{G}) = (\boldsymbol{R} \times \boldsymbol{R}, \mathscr{G})$ are not LP-isomorphic. Nevertheless, there is an obvious LP-monomorphism of $(\boldsymbol{R}^2, \mathscr{F}) \to (\boldsymbol{R}^2, \mathscr{G})$, as indicated in Figure 2.

2. Connexions and metrics

Suppose that (M, Φ) is an AP-manifold, and α is an affine connexion on M. A natural question is the following: is it possible to choose α in such a way that the fields ϕ_i of Φ are all parallel with respect to α? This question has been settled by Walker [15] (see also Willmore [17], [18]) who showed that such an α always exists.

On the other hand, if we ask that α should be the connexion of some Riemannian metric ρ of M, the situation is quite different. Trivially, we can choose ρ in such a way that the fields ϕ_i are mutually orthogonal. For let ξ_i be the m_i-plane bundle with fibre $\phi_i(x)$ at each $x \in M$, and ρ_i be any smooth Riemannian metric for ξ_i, Then a suitable choice of ρ is obtained by putting

$$\rho(\oplus_i X_i, \oplus_i Y_i) = \textstyle\sum_i \rho_i(X_i, Y_i) \ .$$

But there need not exist a Riemannian metric with respect to which each ϕ_i is parallel, as we shall see below. This motivates the notion of *grid*, which we consider next.

3. Grid structures

Let $\mathscr{F}$ be an LP-structure on M, and ρ a smooth Riemannian metric for M. We say that $\mathscr{F}$ and ρ are *compatible* if the foliations $\mathscr{F}_i$ of $\mathscr{F}$ are mutually orthogonal and totally geodesic. This latter condition is equivalent to the requirement that each of the associated fields ϕ_i of tangent m_i-planes be parallel with respect to ρ (see [8], [11] or [14]). A *grid-structure* or *grid* on M is a pair $\Gamma = (\mathscr{F}, \rho)$, where $\mathscr{F}$ is an LP-structure on M, and ρ is a Riemannian metric for M compatible with $\mathscr{F}$. The pair (M, Γ) is called a *grid-manifold*. If (M, Γ) and (N, Δ) are grid-manifolds, where $\Gamma = (\mathscr{F}, \rho)$ and $\Delta = (\mathscr{G}, \sigma)$, then a *grid-morphism* from (M, Γ) to (N, Δ) is an LP-morphism $(f, \zeta)\colon (M, \mathscr{F}) \to (N, \mathscr{G})$ such that $\rho = f^*\sigma$. In particular, f is a smooth immersion. Also if (f, ζ) is a grid-isomorphism, then f is (global) isometry from (M, ρ) onto (N, σ).

In what follows, most grid-morphisms (f, ζ) will be grid-epimorphisms, and ζ is the identity map in these cases. Accordingly, we often abbreviate (f, ζ) to f.

A given LP-structure on M may be compatible with many Riemannian metrics. For example, the standard LP-structure on $\times_i M_i$ is compatible with the product $\times_i \rho_i$ of any Riemannian metrics ρ_i on M_i. On the other hand, the LP-structure on S^3 associated with the Reeb foliation in the manner of Example (3) does not admit a compatible Riemannian metric.

We also observe that an LP-structure may admit a compatible Riemannian metric, and yet fail to admit one which is *complete*. Example (4) illustrates this fact: as we mentioned above, this LP-manifold $(\boldsymbol{R}^2, \mathscr{F})$ can be LP-embedded as an LP-submannifold $(N, \mathscr{G}\,|\,N)$ of the standard LP-manifold $(\boldsymbol{R} \times \boldsymbol{R}, \mathscr{G})$. Referring to Figure 2, we note that a complete compatible metric on N must assign an infinite length to the open line segment joining $(0, 0)$ to $(0, 1)$, and a finite length to that joining $(1, 0)$ to $(1, 1)$. However, compatibility requires that these lengths be equal, which is impossible.

We now describe a method for constructing grid-manifolds. It will appear in the next section that every *complete* grid-manifold is obtained in this way.

Let M_i be a smooth m_i-manifold with a smooth Riemannian metric ρ_i $(i = 1, \cdots, r)$. Then as above $\Gamma = (\mathscr{F}, \rho)$ is a grid of type $(m_1, \cdots, m_r)$ on $M = \times_i M_i$, where $\mathscr{F}$ is the standard LP-structure on M and $\rho = \times_i \rho_i$. Suppose now that G is a discrete group of grid-automorphisms f of (M, Γ), such that the isometries f act freely on M. Then $M' = M/G$ is a smooth manifold having an obvious grid-structure Γ/G of type $(m_1, \cdots, m_r)$, with a grid-epimorphism $(M, \Gamma) \to (M/G, \Gamma/G)$. Moreover, if each M_i is complete, then so is M/G.

We give two examples of this construction to illustrate the theorems which follow.

(7) Let Z_2 act on $M = S^p \times R^q$ by $\zeta(x, y) = (-x, -y)$, where ζ is the nontrivial element of Z_2. If $\mathscr{F} = (\mathscr{F}_1, \mathscr{F}_2)$ is the standard grid on $S^p \times R^q$, where both S^p and R^q have their standard metrics, then Z_2 acts freely on $(M, \mathscr{F})$ as a group of grid-automorphisms. The quotient space $M' = M/Z_2$ has the homotopy-type of real projective k-space P^k. Although M' is not simply-connected, the foils $F_1(Z)$, $F_2(Z)$ at 'most' points of M' are both simply-connected, provided $p > 1$.

(8) Let Z_2 act on $S^p \times S^q$ by $\zeta(x, y) = (-x, y')$, where $y_i = y_i'$ for $y = 1, \cdots, q$ and $y'_{q+1} = -y_{q+1}$. Thus ζ is reflexion in 0 for S^p and reflexion in the equator for S^q. Again $(M', \mathscr{F}') = (S^p \times S^q/Z_2, \mathscr{F}/Z_2)$ is a grid-manifold of type (p, q). In this case, all foils of both foliations of M' are simply-connected (although, of course, $\pi_1(M') = Z_2$ as in (7)).

4. Structure of grid-manifolds

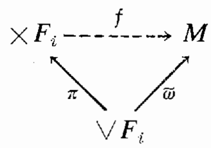
Let $(M, \mathscr{F})$ be an LP-manifold, and $F(x) = (F_1(x), \cdots, F_r(x))$ be the foil-sequence at $x \in M$, and $\vee_i F_i(x) = (+_i F_i(x))/+_i\{x\}$ denote the wedge of $F_1(x), \cdots, F_r(x)$, where $+$ denotes topological sum. The inclusions $\theta_i(x)\colon F_i(x) \subset M$ induce a map $\tilde{\omega}\colon \vee F_i(x) \to M$, and the injections $\pi_i\colon F_i(x) \to \times_{i=1}^{r} F_i(x)$ given by $\pi_i(y) = (y_1, \cdots, y_r)$, where $y_j = x$ $(j \neq i)$ and $y_i = y$, determine an injective map $\pi\colon \vee F_i(x) \to \times F_i(x)$.

Suppose now that ρ is a smooth Riemannian metric on M compatible with $\mathscr{F}$. Then $(\mathscr{F}, \rho) = \Gamma$ is a grid on M, and ρ induces a Riemannian metric $\rho_i(x) = \theta_i^*(x)\rho$ on $F_i(x)$, and hence a standard grid Γ^* on $\times F_i(x)$. We can now state our main theorem. The argument given by de Rham in [8] also serves as a proof of this result. We therefore give only a sketch of the proof, referring to [8] for full details. An alternative approach can be found in [6].

A foil sequence $(F_i, \cdots, F_r)$ is said to be 1-connected if F_i is 1-connected for all $i = 1, \cdots, r$.

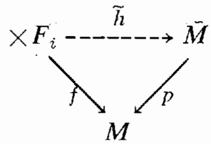
Theorem A. *Let (M, Γ) be a complete grid-manifold, and $(F_1, \cdots, F_r)$ be a 1-connected foil-sequence, and Γ^* denote the grid induced by Γ on $\times F_i$.*

Then there is a grid-epimorphism $f\colon (\times F_i, \Gamma^*) \to (M, \Gamma)$ *such that* $f \circ \pi = \tilde{\omega}$.

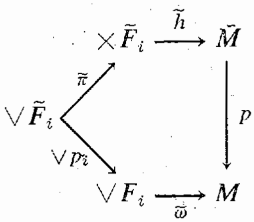


We now list some corollaries of Theorem A; among these is the grid version of de Rham's original theorem. We remark that if $\tilde{M}$ is the universal covering of M, then the projection p induces a grid $\tilde{\Gamma}$ on $\tilde{M}$ from any grid Γ on M.

Corollary 1. *Under the hypothesis of Theorem A, there is a grid-isomorphism* $\tilde{h}\colon (\times F_i, \Gamma^*) \to (\tilde{M}, \tilde{\Gamma})$ *such that* $p \circ \tilde{h} = f$.



Corollary 2. *Let* (M, Γ) *be a complete grid-manifold, and* $(F_i, \cdots, F_r)$ *any foil-sequence. Then there is a grid-isomorphism* $\tilde{h}\colon \times \tilde{F}_i \to \tilde{M}$ *such that* $p \circ \tilde{h} \circ \tilde{\pi} = \tilde{\omega} \circ \vee p_i$, *where* $p_i\colon \tilde{F}_i \to F_i$ *is the universal covering map.*



Corollary 3. *If* (M, Γ) *is a complete grid-manifold of type* $(m_1, \cdots, m_r)$, *then for all* $i = 1, \cdots, r$ *and all* $x, y \in M$, $\tilde{F}_i(x)$ *is (globally) isometric to* $\tilde{F}_i(y)$.

Corollary 4 (*de Rham*). *Any* 1*-connected complete grid-manifold* (M, Γ) *is globally isometric to* $\times F_i$, *where* $(F_1, \cdots, F_r)$ *is any foil-sequence of* Γ.

Corollary 5. *Any complete grid-manifold* (M, Γ) *of type* $(m_1, \cdots, m_r)$ *is grid-isomorphic to* $\left(\left(\times_{i=1}^{r} M_i\right)/G, \Gamma'/G\right)$, *where each* M_i *is a complete Riemannian* m_i*-manifold,* Γ' *is the standard grid, and* G *is a freely acting discrete group of grid-automorphisms.*

Since any such quotient is a grid-manifold, Corollary 5 characterises complete grid-manifolds. We now outline a proof of Theorem A.

Proof of Theorem A. We remark first that in view of Examples (7) and (8) above, M itself need not be 1-connected, and there may be some $y \in M$ and some $i = 1, \cdots, r$ for which $F_i(y)$ is not 1-connected. However, each foil of $\mathscr{F}_i$ is totally geodesic in M and everywhere orthogonal to each foil of $\mathscr{F}_j$ $(i \neq j)$. In fact, for all $y \in M$, $F_i(y)$ is a complete Riemannian m_i-manifold with respect to the 'intrinsic' topology determined by the exponential map of M and the metric $\rho_i(y)$.

Consider the foil-sequence $(F_1, \cdots, F_r)$ at $x \in M$. By definition, there exist open neighbourhoods U_i of x in F_i $(i = 1, \cdots, r)$ and a grid-monomorphism $f_U \colon (\times U_i, \Gamma') \to (M, \Gamma)$ such that $f_U \circ \pi_U = \tilde{\omega}_U$, where $\pi_U \colon \vee U_i \to \times F_i$ and $\tilde{\omega}_U \colon \vee U_i \to M$ are the natural maps, and Γ' is the standard grid.

Now let $\nu \colon I \to \times F_i$ be an arc with $\nu(0) \in U$. De Rham [8, § 6] shows that the grid-monomorphism f_U can be extended to a grid-map $f_U^\nu \colon U_\nu \to M$, where $U_\nu = U \cup W$ and W is some open neighborhood of $\nu(I)$ in $\times F_i$. Here U_ν is given the grid induced from Γ' by the inclusion. Moreover, the induced map $T_y(\times F_i) \xrightarrow{Tf_U^\nu} T_z M$ of tangent spaces, where $y = \nu(1)$ and $z = f_U^\nu(y)$, depends only on the homotopy class of ν (rel. end-points) in $\times F_i$.

It follows that if F_i is 1-connected for all $i = 1, \cdots, n$, then f_U extends uniquely to a grid-map $f \colon \times F_i \to M$, which is an epimorphism in view of the completeness of M and F_i, and $f \circ \pi = \tilde{\omega}$ as required.

5. Irreducible and normal grids

We now express the group $I(M, \rho)$ of isometries of a Riemannian manifold (M, ρ) in terms of certain LP-structures on M compatible with the metric ρ.

Let $LP(M, \rho)$ denote the set of all LP-structures $\mathscr{F}$ on M which are compatible with ρ. We define a partial ordering $\leq$ on $LP(M, \rho)$ by putting $\mathscr{F} \leq \mathscr{G}$ if there is an LP-morphism $(1_M, \zeta) \colon (M, \mathscr{F}) \to (M, \mathscr{G})$. If $\mathscr{F}$ is a minimal element of $(LP(M, \rho), \leq)$, then we say that $\mathscr{F}$ is *irreducible* and that $\Gamma = (\mathscr{F}, \rho)$ is an *irreducible grid* on M. It is shown in [8] that if $\mathscr{F} = (\mathscr{F}_1, \cdots, \mathscr{F}_r)$ and $\mathscr{G} = (\mathscr{G}_1, \cdots, \mathscr{G}_s)$ are irreducible LP-structures on M (compatible with ρ) of types $(m_1, \cdots, m_r)$ and $(n_1, \cdots, n_s)$ respectively, then $r = s$, and there is a permutation $p \colon \{1, \cdots, r\} \to \{1, \cdots, r\}$ such that for all $i = 1, \cdots, r$, $m_i = n_{p(i)}$, and if $m_i > 1$, then $\mathscr{F}_i = \mathscr{G}_{p(i)}$ also. Thus any irreducible LP-structure on (M, ρ) is unique up to the order of its foliations of dimension >1, and up to the choice of its 1-dimensional foliations.

These observations motivate the following definition. We say that $\mathscr{F} = (\mathscr{F}_1, \cdots, \mathscr{F}_r) \in LP(M, \rho)$ of type $(m_1, \cdots, m_r)$ is *normal* on (M, ρ) if $m_i > 1$ for $j = 1, \cdots, r-1$ and there is an irreducible LP-structure $\mathscr{F}' = (\mathscr{F}_1, \cdots, \mathscr{F}_{r-1}, \mathscr{F}'_1, \cdots, \mathscr{F}'_{m_r})$ (necessarily of type $(m_1, \cdots, m_{r-1}, 1, \cdots, 1)$) such that $\mathscr{F}' \leq \mathscr{F}$, or if $\mathscr{F}$ itself is irreducible and $m_i > 1$ for all $i = 1, \cdots, r$.

Remark. In the former case, each foil $F_r(x)$ of $\mathscr{F}_r$ is a *flat* m_r-manifold, and $\tilde{F}_r(x) = \boldsymbol{R}^{m_r}$. For further details of such spaces, see [1], [2] or [19].

If $\mathscr{F}$ is normal on (M, ρ), then we say that $(\mathscr{F}, \rho)$ is a *normal grid* on M. We denote the set of normal LP-structures on (M, ρ) by $\mathscr{N}(M, \rho)$. Now the isometry group $I(M, \rho)$ of (M, ρ) acts on $LP(m, \rho)$ as a group of permutations, as follows. Let $\mathscr{F} = (\mathscr{F}_1, \cdots, \mathscr{F}_r) \in LP(M, \rho)$, and let $g \in I(M, \rho)$. Then $g \cdot \mathscr{F} = \mathscr{G} = (\mathscr{G}_1, \cdots, \mathscr{G}_r) \in LP(M, \rho)$, where for all $x \in M$, $G_i(g(x)) = g(F_i(x))$. Thus the set $\mathscr{N}(M, \rho)$ of normal LP-structures on (M, ρ) is setwise invariant under this action of $I(M)$. Further, if $g \in I(M)$ and $\mathscr{F} \in \mathscr{N}(M, \rho)$ are such that $g \cdot \mathscr{F} = \mathscr{F}$, then $g \cdot \mathscr{F}' = \mathscr{F}'$ for *all* $\mathscr{F}' \in \mathscr{N}(M, \rho)$. Hence the group of all grid-automorphisms of any normal grid $\Gamma = (\mathscr{F}, \rho)$ on M is independent of $\mathscr{F} \in \mathscr{N}(M, \rho)$, and coincides with the subgroup $I'(M)$ of $I(M)$ which acts trivially on $\mathscr{N}(M, \rho)$. We observe that $I'(M)$ is a normal subgroup of $I(M)$.

To sum up, the isometry group $I(M, \rho)$ of any Riemannian manifold is the group of all grid-isomorphisms between normal grid-manifolds (M, Γ), where Γ is of the form $(\mathscr{F}, \rho)$. In particular, if there is a normal grid $\Gamma = (\mathscr{F}, \rho)$ on M of type $(m_1, \cdots, m_r)$, where $m_i \neq m_j$ for all $i \neq j$, then every isometry of (M, ρ) is a grid-automorphism of (M, Γ).

6. Related structures

The special class of grids $((\mathscr{F}_1, \mathscr{F}_2), \rho)$ on (M, ρ) for which at least one of the foliations $\mathscr{F}_i$ *fibres* M have been studied by Walker [14] and Reinhart [9] (see also Hermann [4]). We mention some other natural lines of development.

One may try to extend the above ideas to pseudo-Riemannian manifolds, studying parallel fields of tangent planes on these spaces. Unfortunately, the local-product structures on which the theory of grids is based do not survive this generalisation. A local analysis of the situation has been made by Walker [12], [13], while Wu [20] has obtained a partial extension of de Rham's theorem to the pseudo-Riemannian case. An 'affine analogue' of the de Rham theorem appears in [5].

Again, suppose that (M, Γ) is a grid-manifold, with $\Gamma = (\mathscr{F}, \rho)$ and $\mathscr{F} = (\mathscr{F}_1, \cdots, \mathscr{F}_r)$. Let G be a discrete group of freely-acting isometries of M such that for all $g \in G$, all $x \in M$ and all $i = 1, \cdots, r$, $g(F_i(x)) = F_j(x)$ for some $j = 1, \cdots, r$. Then M/G has a natural Riemannian structure, and each foil F_i of each $\mathscr{F}_i$ is immersed in M/G by the projection $M \to M/G$ as a totally geodesic *immersed* submanifold whose self-intersections are either tangent or normal.

The resulting structure is called a *twisted grid* on M/G. A simple illustration of this idea may be described as follows.

Consider the discrete group G of isometries of the standard flat grid manifold $\boldsymbol{R} \times \boldsymbol{R}$, generated by the translation P and the glide-reflexion Q given by $P(x, y) = (x, y + 1)$ and $Q(x, y) = (x + 1, -y)$. Thus $(\boldsymbol{R} \times \boldsymbol{R})/G$ is isometric to the standard flat Klein bottle K. The grid on $\boldsymbol{R} \times \boldsymbol{R}$ is not equivariant under the projection $\boldsymbol{R} \times \boldsymbol{R} \to K$, but carries down to a twisted grid on K. This produces a situation in which K is covered by a family of parallel lines each of which is also orthogonal to every member of the family, including itself (see Figure 3). By using the group generated by Q alone, one gets an analogous

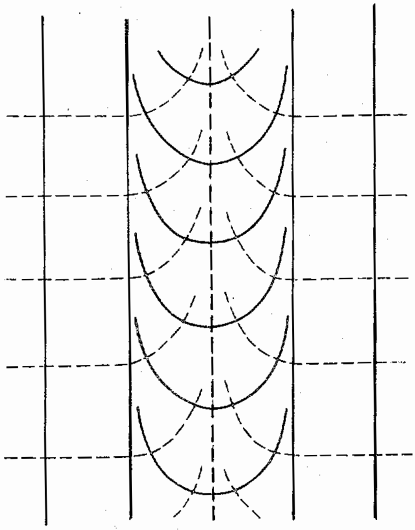


Fig. 1

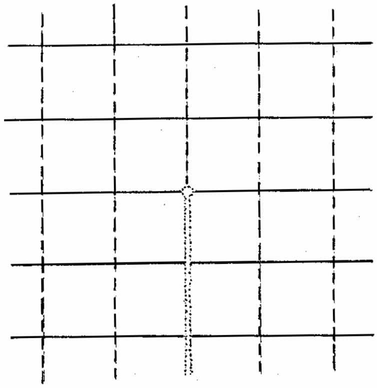


Fig. 2

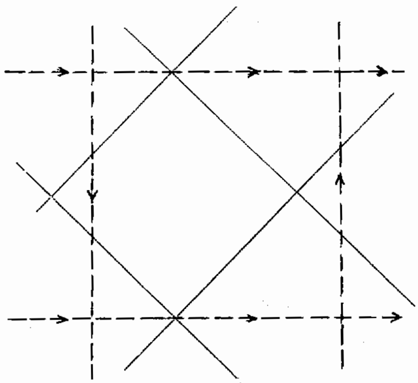


Fig. 3. A twisted grid on the flat Klein bottle

situation on the Möbius band. Twisted grids will be studied in more detail in [10].

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